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Thermoelastic characteristics of a composite solid with initial stresses

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Abstract

Thermoelastic problem for a composite solid with initial stresses is considered on the basis of the asymptotic homogenization method. The homogenized model is constructed by means of the two-scale asymptotic homogenization techniques. The major result of a present paper is that the effective (homogenized) thermoelastic characteristics of the composite material depend not only on local distributions of all types of material characteristics: local elastic properties, local thermoelastic properties, but also on local initial stresses. Therefore it is shown that for the inhomogeneous (composite) material local initial stresses contribute towards values of the effective characteristics of the material. This kind of interaction is not possible for the homogeneous materials. From the mathematical viewpoint, the asymptotic homogenization procedure is equivalent to the computation of G-limit of the corresponding operator. And the above noted phenomenon is based on the fact that in the considering case the G-limit of a sum is not equal to the sum of G-limits. The developed general homogenized model is illustrated in the particular case of the small initial stresses, which is common for the practical mechanical problems. The explicit formulas for the effective thermoelastic characteristics and numerical results are obtained for a laminated composite solid with the initial stresses.
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1. Homogenization for stressed inhomogeneous media

Consider an inhomogeneous (composite) elastic solid of a periodic structure with a periodicity cell P_ε shown in Fig. 1. Here parameter $\varepsilon \ll 1$ denotes a characteristic dimension of the periodicity cell. The condition $\varepsilon \ll 1$ is formalized as $\varepsilon \to 0$.

The solid is subject to forces $\mathbf{F}(\mathbf{x})$ that cause stresses $\sigma^{\varepsilon}_{ij}(0)$. These stresses are called the initial stresses. By applying an additional force $\mathbf{f}(\mathbf{x})$ and a temperature change $\Theta(\mathbf{x})$, the problem of deformation of a body having initial stresses arises. The general description of a solid with initial stresses has been considered in Washizu (1982). The following problems have been formulated to describe the basic (initial) state:

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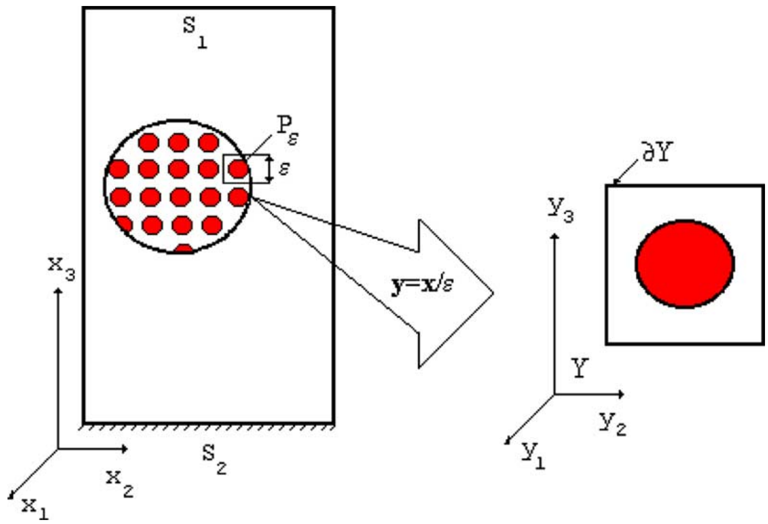

Fig. 1. Composite solid of a periodic structure and the periodicity cell Y in the “fast” variables $\mathbf{y}$.

$$\boldsymbol{L}_\varepsilon(0,0)\mathbf{v}^\varepsilon = \mathbf{F} \quad \text{in } Q, \quad \sigma_{ij}^\varepsilon(0)n_j^\varepsilon = G_i(\mathbf{x}) \quad \text{on } S_1, \quad \mathbf{v}^\varepsilon = \mathbf{v}^0 \quad \text{on } S_2 \tag{1.1}$$

and to determine the additional displacements:

$$\boldsymbol{L}_\varepsilon(\sigma,\Theta)\mathbf{u}^\varepsilon = \mathbf{f} \quad \text{in } Q, \quad \sigma_{ij}^\varepsilon(\sigma)n_j^\varepsilon = g_i(\mathbf{x}) \quad \text{on } S_1, \quad \mathbf{u}^\varepsilon = \mathbf{u}^0 \quad \text{on } S_2, \tag{1.2}$$

where $\mathbf{n}^\varepsilon$ is the normal to S_1, S_1 and S_2, are shown in Fig. 1. Summation with respect to the repeating indices is assumed here and in the sequel.

In the above Eqs. (1.1) and (1.2) $\mathbf{v}^\varepsilon$ and $\mathbf{u}^\varepsilon$ are the initial and additional displacements, respectively; $c_{ijkl}(\mathbf{x}/\varepsilon)$ are the local elastic properties of the material with no initial stresses;

$$\sigma_{ij}^\varepsilon(0) = c_{ijkl}(\mathbf{x}/\varepsilon)\partial v_k^\varepsilon/\partial x_l \tag{1.3}$$

are the initial stresses;

$$\sigma_{ij}^\varepsilon(\sigma) = (c_{ijkl}(\mathbf{x}/\varepsilon) + \sigma_{jl}^\varepsilon(0)\delta_{ik})\partial u_k^\varepsilon/\partial x_l + t_{ij}\Theta \tag{1.4}$$

are the so-called additional stresses (see Washizu, 1982).

$\boldsymbol{L}_\varepsilon(\sigma,\Theta)\mathbf{u} = \partial/\partial x_j[(c_{ijkl}(\mathbf{x}/\varepsilon) + \sigma_{jl}^\varepsilon(0)\delta_{ik})\partial u_k/\partial x_l + t_{ij}(\mathbf{x}/\varepsilon)\Theta]$ is the thermoelastic operator that also incorporates the initial stresses;

$\delta_{ik} = 1$, if $i = k$, and $\delta_{ik} = 0$, if $i \neq k$;

$t_{ij}(\mathbf{x}/\varepsilon) = -c_{ijkl}(\mathbf{x}/\varepsilon)\alpha_{kl}(\mathbf{x}/\varepsilon)$, where $\alpha_{kl}(\mathbf{x}/\varepsilon)$ are the coefficients of the linear thermal expansion;

$\boldsymbol{L}_\varepsilon(0,\Theta) = \partial/\partial x_j[c_{ijkl}(\mathbf{x}/\varepsilon)\partial u_k/\partial x_l + t_{ij}(\mathbf{x}/\varepsilon)\Theta]$ is the thermoelastic operator with no initial stresses;

$\boldsymbol{L}_\varepsilon(0,0)\mathbf{u} = \partial/\partial x_j[c_{ijkl}(\mathbf{x}/\varepsilon)\partial u_k/\partial x_l]$ is the elastic operator with no initial stresses.

The functions $c_{ijkl}(\mathbf{x}/\varepsilon)$, $\sigma_{jl}^\varepsilon(0)(\mathbf{x},\mathbf{x}/\varepsilon)$, $t_{ij}(\mathbf{x}/\varepsilon)$, $\alpha_{kl}(\mathbf{x}/\varepsilon)$, are periodic in variables $\mathbf{x}$ with periodicity cell P_ε.

Let us describe the distinctive features of the problem under consideration. The operator $\boldsymbol{L}_\varepsilon(\sigma,\Theta)$ can be written in the form

$$\boldsymbol{L}_\varepsilon(\sigma,\Theta)\mathbf{u} = \boldsymbol{L}_\varepsilon(0,0)\mathbf{u} + \boldsymbol{m}_\varepsilon\mathbf{u} + \partial/\partial x_j[t_{ij}(\mathbf{x}/\varepsilon)\Theta], \tag{1.5}$$

where $\boldsymbol{L}_\varepsilon(0,0)$ is the elastic operator (see above), the operator $\boldsymbol{m}_\varepsilon$ defined as $\boldsymbol{m}_\varepsilon \mathbf{u} = \partial/\partial x_j[\sigma^\varepsilon_{jl}(0)(\mathbf{x},\mathbf{x}/\varepsilon)\delta_{ik} \times \partial u_k/\partial x_l]$ incorporates the initial stresses and the term $\partial/\partial x_j[t_{ij}(\mathbf{x}/\varepsilon)\Theta]$ incorporates the thermal expansion.

It is known, see e.g., Bensoussan et al. (1978), that composite body as $\varepsilon \to 0$ can be replaced by a homogeneous (referred to as ''homogenized'') body similar to it in mechanical behavior. This fact is known for elastic solids, see e.g., Duvaut (1976), for thermoelastic solids, see e.g., Kolpakov (1980), Kalamkarov (1989), and for elastic solids with initial stresses see Kolpakov (1989). From the mathematical viewpoint, the homogenization procedure is equivalent to the computation of a G-limit of a corresponding operator (see Marcellini, 1975). We would like to compute the homogenized (also called effective) characteristics of the body. Thus we would like to compute G-limit of the operator $\boldsymbol{L}_\varepsilon(\sigma,\Theta)$ (1.5) as $\varepsilon \to 0$. Commonly, ''limit of a sum is equal to the sum of limits''. But for G-limits that is not true, and it is possible that ''G-limit of a sum is not equal to the sum of G-limits'' (see Marcellini, 1975). From the mechanical viewpoint it is explained by the occurrence of a general state of local stress and strain when the uniform homogenized stresses are applied to an inhomogeneous medium (see Kolpakov, 2001). The case when ''G-limit of a sum is not equal to the sum of G-limits'' also occurs for the sum of a differential operator and a fast oscillating function of large amplitude, see Flery et al. (1979), or derivative of a fast oscillating function (see Kolpakov, 1980; Kalamkarov and Kolpakov, 1997).

In the case considered in the present paper we deal with the homogenization problem for the operator (1.5), which is the sum of the operators $\boldsymbol{L}_\varepsilon(0,0)\mathbf{u} = \partial/\partial x_j[c_{ijkl}(\mathbf{x}/\varepsilon)\partial u_k/\partial x_l]$ and $\boldsymbol{m}_\varepsilon(\sigma)\mathbf{u} = \partial/\partial x_j[\sigma^\varepsilon_{jl}(0)\delta_{ik}\partial u_k/\partial x_l]$ of the same order with the fast oscillating coefficients and the derivative of the fast oscillating function $\partial/\partial x_j(t_{ij}(\mathbf{x}/\varepsilon)\Theta)$. For all of them ''G-limit of a sum is not equal to the sum of G-limits''. It is the first characteristic feature of the problem under study.

Another distinctive feature of the problem is related to the effect of ''lose of symmetry'' in the problem for elastic body with initial stresses Kolpakov (2004). The elastic constants c_{ijkl} have well-known symmetries, in particular $c_{ijkl} = c_{jikl}$ and $c_{ijkl} = c_{klij}$ (see e.g., Timoshenko and Goodier, 1970). Introducing $B_{ijkl} = c_{ijkl} + \sigma^\varepsilon_{jl}(0)\delta_{ik}$, we can rewrite (1.4) in the form $\sigma^\varepsilon_{ij}(\sigma) = B_{ijkl}\,\partial u^\varepsilon_k/\partial x_l + t_{ij}(\mathbf{x}/\varepsilon)\Theta$ and consider it as a constitutive equation for a stressed thermoelastic body. In contrast to the elastic constants, the quantities B_{ijkl} do not have all symmetries common for the elastic constants. Namely $B_{ijkl} \neq B_{ijlk}$. At the same time the quantities B_{ijkl} retain the symmetry $B_{ijkl} = B_{klij}$. Indeed, $B_{ijkl} = c_{ijkl} + \sigma^\varepsilon_{jl}(0)\delta_{ik} = c_{klij} + \sigma^\varepsilon_{lj}(0)\delta_{ki} = B_{klij}$ because c_{ijkl} and $\sigma^\varepsilon_{jl}(0)\delta_{ik}$ are symmetric with respect to the change of the indices $i \leftrightarrow k$ $j \leftrightarrow l$.

1.1. Asymptotic homogenization method applied to the thermoelastic composites with initial stresses

It will be shown in this subsection that the thermoelastic composite body with initial stresses as $\varepsilon \to 0$ can be replaced by a homogeneous (referred to as ''homogenized'') body similar to it in mechanical behavior and the solutions of the problems (1.1) and (1.2) may be approximated by the solutions of the so-called homogenized problems:

$$\boldsymbol{L}(0,0)\mathbf{v} = \mathbf{F} \quad \text{in } Q, \quad \sigma_{ij}(0)n_j = 0 \quad \text{on } S_1, \quad \mathbf{v} = 0 \quad \text{on } S_2, \tag{1.6}$$

$$\boldsymbol{L}(\sigma,\Theta)\mathbf{u} = \mathbf{f} \quad \text{in } Q, \quad \sigma_{ij}(\sigma)n_j = 0 \quad \text{on } S_1, \quad \mathbf{u} = 0 \quad \text{on } S_2. \tag{1.7}$$

Here: $\mathbf{v}$ and $\mathbf{u}$ are the ''homogenized'' displacements (that is, the displacements determined from the homogenized problems);

- $\boldsymbol{L}(0,\Theta)\mathbf{v} = \partial/\partial x_j[a_{ijkl}(0)\partial v_k/\partial x_l + T_{ij}(0)\Theta]$ is the homogenized operator corresponding to (1.6) (the homogenization of thermoelastic problem with no initial stresses);
- $\boldsymbol{L}(\sigma,\Theta)\mathbf{u} = \partial/\partial x_j[a_{ijkl}(\sigma)\partial u_k/\partial x_l + T_{ij}(\sigma)\Theta]$ is the homogenized operator corresponding to (1.7) (the homogenization with initial stresses);

- $a_{ijkl}(0)$ are the coefficients of operator $\boldsymbol{L}(0,0)$ (these are the "homogenized" elastic constants of the body with no initial stresses) and $T_{ij}(0)$ are the "homogenized" thermoelastic constants of the body with no initial stresses;
- $a_{ijkl}(\sigma)$ are the coefficients of operator $\boldsymbol{L}(\sigma)$ (these are the "homogenized" constants of the body with initial stresses) and $T_{ij}(\sigma)$ are the "homogenized" thermoelastic constants of the body with initial stresses;

$$\sigma_{ij}(\sigma,\Theta) = a_{ijkl}(\sigma)\partial u_k/\partial x_l + T_{ij}(\sigma)\Theta;$$

$$\sigma_{ij}(0,\Theta) = a_{ijkl}(0)\partial v_k/\partial x_l + T_{ij}(0)\Theta;$$

$\langle\cdot\rangle = (\text{mes Y})^{-1}\int_Y \mathbf{dy}$ is the average value over the periodicity cell $Y = \varepsilon^{-1}P_\varepsilon = \{\mathbf{y} = \mathbf{x}/\varepsilon : \mathbf{x} \in P_\varepsilon\}$ in the "fast" variables $\mathbf{y} = \mathbf{x}/\varepsilon$ (see Fig. 1).

Number of authors (see Bensoussan et al., 1978; Sanchez-Palencia, 1980; Bakhvalov and Panasenko, 1989; Oleinik et al., 1990; Cioranescu and Saint Jean Paulin, 1979; Kalamkarov, 1992, see also references in the above books) presented the homogenization procedures for an elastic body with no initial stresses. From the references above it is known that $\sigma_{ij}(0)$ are equal to average value of the initial stresses:

$$\sigma_{ij}(0) = \langle\sigma^{\varepsilon}_{ij}(0)\rangle. \tag{1.8}$$

The homogenization problem for thermoelastic body was analyzed by Kolpakov (1980) (see also Kalamkarov and Kolpakov (1997)). Kolpakov (1980) has shown that Eq. (1.8) remains valid for thermoelastic problem.

1.2. Computation of homogenized constants of a stressed thermoelastic body

To derive formulas for computing the homogenized constants of stressed body we use the two-scale asymptotic expansion method (see e.g., Bakhvalov and Panasenko, 1989). We use the following asymptotic expansions:

Expansion for displacements

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^{(0)}(\mathbf{x}) + \varepsilon\mathbf{u}^{(1)}(\mathbf{x},\mathbf{y}) + \cdots = \mathbf{u}^{(0)}(\mathbf{x}) + \sum_{k=1}^{\infty}\varepsilon^k\mathbf{u}^{(k)}(\mathbf{x},\mathbf{y}). \tag{1.9}$$

Expansion for stresses

$$\sigma^{\varepsilon}_{ij}(\sigma) = \sum_{k=0}^{\infty}\varepsilon^k\sigma^{(k)}_{ij}(\mathbf{x},\mathbf{y}). \tag{1.10}$$

Here $\mathbf{x}$ are the "slow" variables, and $\mathbf{y} = \mathbf{x}/\varepsilon$ are the "fast" variables. The functions in the right-hand side of (1.9) and (1.10) are assumed to be periodic in $\mathbf{y}$ with periodicity cell Y. Note that the term $\mathbf{u}^{(0)}(\mathbf{x})$ in (1.9) depends on the "slow" variable $\mathbf{x}$ only.

With the use of two-scale expansions, the differential operators are presented in the form of sum of operators in $\mathbf{x}$ and in $\mathbf{y}$ (see e.g., Bensoussan et al., 1978). For the function $Z(\mathbf{x},\mathbf{y})$ of the arguments $\mathbf{x}$ and $\mathbf{y}$, as in the right-hand sides of (1.9) and (1.10), this representation takes the form

$$\partial Z/\partial x_i = Z_{,ix} + \varepsilon^{-1}Z_{,iy}. \tag{1.11}$$

Here and below the subscribe, *ix* means $\partial/\partial x_i$ and, *iy* means $\partial/\partial y_i$.

Substituting (1.9) and (1.10) into Eq. (1.4), we obtain with allowance for the differentiating rule (1.11)

$$\sum_{k=0}^{\infty} \varepsilon^k \sigma_{ij}^{(k)} = \sum_{k=0}^{\infty} \varepsilon^k B_{ijmn}(u_{m,nx}^{(k)} + \varepsilon^{-1} u_{m,ny}^{(k)}) + t_{ij}\Theta, \quad k = 0, 1, \ldots, \tag{1.12}$$

where

$$B_{ijmn} = c_{ijmn} + \sigma_{jn}^{\varepsilon}(0)\delta_{im}. \tag{1.13}$$

Equating the terms with identical powers of ε in (1.12), we obtain

$$\sigma_{ij}^{(0)} = B_{ijmn} u_{m,nx}^{(0)} + B_{ijmn} u_{m,ny}^{(1)} + t_{ij}\Theta,$$

$$\sigma_{ij}^{(k)} = B_{ijmn} u_{m,nx}^{(k)} + B_{ijmn} u_{m,ny}^{(k+1)}, \quad k = 1, 2, \ldots \tag{1.14}$$

The equilibrium equation (see (1.2) and (1.4) and definition of the operator $\boldsymbol{L}_\varepsilon(\sigma, \Theta)$) may be written for stresses

$$\partial \sigma_{ij}^{\varepsilon}(\sigma)/\partial x_j = f_i \quad \text{in } Q, \quad \sigma_{ij}^{\varepsilon}(\sigma) n_j = 0 \quad \text{on } S_1. \tag{1.15}$$

Substituting (1.10) into the equilibrium equations (1.15), we obtain with account of rule of differentiation (1.11)

$$\sum_{k=0}^{\infty} \varepsilon^k \sigma_{ij,jx}^{(k)} + \sum_{k=0}^{\infty} \varepsilon^{k-1} \sigma_{ij,jy}^{(k)} = f_i \quad \text{in } Q, \quad \sum_{k=0}^{\infty} \varepsilon^k \sigma_{ij}^{(k)} n_j = 0 \quad \text{on } S_1. \tag{1.16}$$

Equating the terms with identical power of ε in (1.16), we obtain an infinite sequence of equations:

$$\sigma_{ij,jx}^{(0)} + \sigma_{ij,jy}^{(1)} = f_i \quad \text{and} \quad \sigma_{ij,jx}^{(k)} + \sigma_{ij,jy}^{(k+1)} = 0 \quad \text{for } k > 0 \text{ in } Y; \quad k = 0, 1, \ldots \tag{1.17}$$

Averaging (1.17) over the periodicity cell Y, we obtain an infinite sequence of the homogenized equilibrium equations, the first of which is the following

$$\langle \sigma_{ij}^{(0)} \rangle_{,jx} = f_i \tag{1.18}$$

Here we use equality $\langle \sigma_{ij,jy}^{(1)} \rangle = 0$, which follows from the formula

$$\int_Y \sigma_{ij,jy}^{(1)} \, d\mathbf{y} = \int_{\partial Y} \sigma_{ij}^{(1)} n_j \, d\mathbf{y} + \int_{\Gamma} \sigma_{ij}^{(1)} n_j \, d\mathbf{y}.$$

The first integral is equal to zero by virtue of periodicity $\sigma_{ij}^{(1)}$ and anti-periodicity vector-normal $\mathbf{n}$. The second integral is equal to zero by virtue of condition $\sigma_{ij}^{(1)} n_j = 0$ on Γ.

Let us consider the problem (1.13), $k = 0$. It can be written as

$$(B_{ijmn}(\mathbf{y}) u_{m,ny}^{(1)} + B_{ijkl}(\mathbf{y}) u_{m,nx}^{(0)} + t_{ij}(\mathbf{y})\Theta)_{,j} = 0 \quad \text{in } Y. \tag{1.19}$$

Allowing for the fact that the function of the argument $\mathbf{x}$ plays the role of a parameter in the problems in the variables $\mathbf{y}$ and $\mathbf{u}^{(0)}$ and Θ depend on $\mathbf{x}$, only, solution of the problem (1.19) with the periodicity conditions can be found in the form

$$\mathbf{u}^{(1)} = \boldsymbol{N}^{mn}(\mathbf{y}) u_{m,nx}^{(0)}(\mathbf{x}) + \boldsymbol{N}^0(\mathbf{y})\Theta(\mathbf{x}) + \mathbf{V}(\mathbf{x}). \tag{1.20}$$

Here $V(\mathbf{x})$ is an arbitrary function of the argument $\mathbf{x}$,which does not influence the final equations, and the periodic function $\boldsymbol{N}^{kl}(\mathbf{y})$ and $\boldsymbol{N}^0(\mathbf{y})$ represent solutions of the following unit cell problems:

The elasticity cellular problem for stressed body:

$$(B_{ijmn}(\mathbf{y}) N_{m,ny}^{kl} + B_{ijkl}(\mathbf{y}))_{,j} = 0 \quad \text{in } Y. \tag{1.21}$$

$\boldsymbol{N}^{kl}(\mathbf{y})$ is periodic in $\mathbf{y}$ with the periodicity cell Y.

The thermoelasticity cellular problem for stressed body:

$$(B_{ijmn}(\mathbf{y})N^0_{m,ny} + t_{ij}(\mathbf{y}))_{,j} = 0 \quad \text{in } Y. \tag{1.22}$$

$\boldsymbol{N}^0(\mathbf{y})$ is periodic in $\mathbf{y}$ with the periodicity cell Y.

Remark. The cellular problems (1.21) and (1.22) look similar to the classical elastic and thermoelastic unit cell problems (see e.g., Bensoussan et al., 1978; Kalamkarov and Kolpakov, 1997). Nevertheless, there is a substantial difference between the problems (1.21) and (1.22) and the classical cellular problems. The coefficients of the classical cellular problems are the elastic constants c_{ijmn}. The coefficients B_{ijmn} of the cellular problems (1.21) and (1.22) are the combination of the elastic constants and the initial stresses. We will use this fact in the next sections of the paper.

Substituting (1.20) into (1.14), we have

$$\sigma^{(0)}_{ij} = (B_{ijmn}(\mathbf{y})N^{kl}_{m,ny} + B_{ijkl}(\mathbf{y}))u^{(0)}_{k,lx}(\mathbf{x}) + (t_{ij}(\mathbf{y}) + B_{ijmn}(\mathbf{y})N^0_{m,ny})\Theta(\mathbf{x}). \tag{1.23}$$

Averaging (1.23) over the cell Y, we obtain the following homogenized constitutive equation

$$\langle\sigma^{(0)}_{ij}\rangle = a_{ijkl}(\sigma)u^{(0)}_{k,lx}(\mathbf{x}) + T_{ij}(\sigma)\Theta(\mathbf{x}), \tag{1.24}$$

where

$$a_{ijkl}(\sigma) = \langle B_{ijmn}(\mathbf{y})N^{kl}_{m,ny} + B_{ijkl}(\mathbf{y})\rangle, \tag{1.25}$$

$$T_{ij}(\sigma) = \langle t_{ij}(\mathbf{y}) + B_{ijmn}(\mathbf{y})N^0_{m,ny}\rangle \tag{1.26}$$

are called the homogenized (or effective) elastic and thermoelastic characteristics of the initially stressed solid.

We introduce the homogenized constants of thermal expansion as $A_{ij}(\sigma) = -c_{ijkl}(\sigma)^{-1}T_{kl}(\sigma)$, where $c_{ijkl}(\sigma)^{-1}$ is the inverse to the tensor $c_{ijkl}(\sigma)$.

1.3. The homogenized model

The homogenized equilibrium equation (1.18), the homogenized constitutive equation (1.24) and the boundary conditions

$$\mathbf{u}^{(0)}(\mathbf{x}) = 0 \quad \text{on } S_2, \quad \sigma^{(0)}_{ij}n^{\varepsilon}_j = 0 \quad \text{on } S_1 \tag{1.27}$$

represent the homogenized problem for stressed body. Substituting (1.25) into (1.24), we can write the homogenized problem in the form (1.7).

The fundamental difference of this problem from the homogenized problem for body having no initial stresses is the dependence of the cellular problems (1.21) and (1.22) and the homogenized coefficients $a_{ijkl}(\sigma)$, $T_{ij}(\sigma)$ on the initial stresses.

2. The case of small initial stresses

Consider the case when the initial stresses $\sigma^{\varepsilon}_{ij}(0)$ are small as compared with the elastic constants c_{ijkl}, and B_{ijkl} (1.13) can be represented as $B_{ijkl}(\mathbf{x},\mathbf{y}) = c_{ijkl}(\mathbf{y}) + \mu b_{ijkl}(\mathbf{x},\mathbf{y})$, cf. (2.1). We will find solution of the cellular problem (1.22) in the form

$$\boldsymbol{N}^0(\mathbf{y}) = \boldsymbol{N}^{00}(\mathbf{y}) + \mu\boldsymbol{N}^{10}(\mathbf{y}) + \cdots = \sum_{s=0}^{\infty} \mu^s \boldsymbol{N}^{s0}(\mathbf{y}). \tag{2.1}$$

All the functions $\boldsymbol{N}^{s0}(\mathbf{y})$ in (2.1) are assumed to be periodic in $\mathbf{y}$ with the periodicity cell Y. Substituting (2.1) into (1.22) and equating the terms with identical powers of μ, we obtain an infinite sequence of problems, the first two of which are the following:

$$(c_{ijnm}(\mathbf{y})N^{00}_{m,ny} + t_{ij}(\mathbf{y}))_{,jy} = 0 \quad \text{in } Y, \tag{2.2}$$

$$(c_{ijnm}(\mathbf{y})N^{10}_{m,ny} + b_{ijmn}(\mathbf{x},\mathbf{y})N^{00}_{m,ny}(y))_{,jy} = 0 \quad \text{in } Y, \tag{2.3}$$

$$\boldsymbol{N}^{10}(\mathbf{y}), N^{00}(\mathbf{y}) \text{ are periodic with the periodicity cell } Y. \tag{2.4}$$

The problem (2.2) is the well-known cellular problem for thermoelastic body with no initial stresses, see Kalamkarov and Kolpakov (1997), or, that is the same, the cellular problem (1.22) with $B_{ijkl} = c_{ijkl}$. Denote solution of the cellular problem (2.2) with no initial stresses by $\mathbf{N}^0(\mathbf{y})$.

Substituting (2.1) into (1.22) and saving the terms up to the linear (in μ) term, we obtain

$$T_{ij}(\sigma) = T_{ij}(0) + \mu n_{ij}(\sigma) + \cdots = \tag{2.5}$$

where $\cdots$ means the terms having the order of μ^2 and higher,

$$\begin{aligned} T_{ij}(0) &= \langle t_{ij} - c_{klmn}N^{0ij}_{k,ly}N^{00}_{m,ny}\rangle, \\ n_{ij}(\sigma) &= \langle -b_{klmn}N^{0ij}_{k,ly}N^{00}_{m,ny} - c_{klmn}N^{1ij}_{k,ly}N^{00}_{m,ny} - c_{klmn}N^{1ij}_{m,ny}N^{10}_{m,ny}\rangle. \end{aligned} \tag{2.6}$$

The coefficients $T_{ij}(0)$ are the homogenized (or effective) thermoelastic constants of the composite material with no initial stresses, see Kolpakov (1980), Kalamkarov and Kolpakov (1997).

$$n_{ij}(\sigma) = \langle \sigma^{\varepsilon}_{ln}(0)N^0_{p,ny}N^{ij}_{p,ly}\rangle. \tag{2.7}$$

Remark. The $n_{ij}(\sigma)$ are expressed in terms of derivatives of $\mathbf{N}^{\alpha\beta}$ and $\mathbf{N}^0$ and cannot be expressed in terms of deformations corresponding to $\mathbf{N}^{\alpha\beta}$ and $\mathbf{N}^0$ in the general case.

The formula (2.7) can be written in terms of the homogenized stresses. According to Oleinik et al. (1990), the local stresses in a body with no initial stresses are given by the formula

$$\sigma^{\varepsilon}_{ij}(0) = c_{ijkl}(\mathbf{x}/\varepsilon)(e_{kl} + N^{pq}_{k,ly}(\mathbf{x}/\varepsilon)e_{pq}) = c_{ijkl}(\mathbf{x}/\varepsilon)(\delta_{kp}\delta_{lq} + N^{pq}_{k,ly}(\mathbf{x}/\varepsilon))\boldsymbol{J}_{pqmn}\sigma_{mn}(0), \tag{2.8}$$

where $e_{pq} = 1/2(\partial v_p/\partial x_q + \partial v_q/\partial x_p)$ are the homogenized strains and $\{\boldsymbol{J}_{pqmn}\} = \{a_{ijkl}(0)\}^{-1}$ is the homogenized compliance tensor.

Substituting last expression from (2.8) into (2.7) in place of $\sigma^{\varepsilon}_{ij}$, we obtain the following formula:

$$T_{ij}(\sigma) = T_{ij}(0) + \mu r_{ijrs}(\sigma)\sigma_{rs}(0), \tag{2.9}$$

where

$$r_{ijrs}(\sigma) = \langle c_{qtcd}N^{rs}_{c,dy}N^0_{p,ty}N^{ij}_{p,qy} + c_{qtrs}N^0_{p,ty}N^{ij}_{p,qy}\rangle \tag{2.10}$$

with summation with respect to the repeating subscripts.

3. Properties of the thermoelastic constants of a stressed composite solid

As seen from Eq. (1.25) the homogenized constants $a_{ijkl}(\sigma)$ depend only on the local elastic constants and initial stresses. The homogenized thermoelastic constants $T_{ij}(\sigma)$ depend on all local characteristics: the local thermoelastic constants, the local elastic constants and also on the local initial stresses.

In the general case, see Kolpakov (1989, 1992), see also comment on the inequality (3.1) in Kolpakov (2001),

$$a_{ijkl}(\sigma) \neq a_{ijkl}(0) + \sigma_{jl}(0)\delta_{ik}, \tag{3.1}$$

$$b_{ij}(\sigma) \neq \langle b_{ij} \rangle, \quad \alpha_{ij}(\sigma) \neq \langle \alpha_{ij} \rangle. \tag{3.2}$$

It is possible to express the effective thermoelastic constants through the solution of the elastic cellular problems only. Indeed, the following equality takes place

$$T_{ij}(\sigma) = \langle t_{ij}(\mathbf{y}) + B_{ijmn}(\mathbf{y})N^0_{m,ny} \rangle = \langle t_{ij}(\mathbf{y}) + t_{mn}(\mathbf{y})N^{ij}_{m,ny} \rangle. \tag{3.3}$$

The above equality (3.3) can be proved as follows. Multiplying (1.21) by N_i^0 and (1.22) by N_i^{kl} and integrating by parts, and applying the periodicity condition, we obtain

$$\langle B_{ijmn}(\mathbf{y})N^{kl}_{m,ny}N^0_{i,jy} + B_{ijkl}(\mathbf{y})N^0_{i,jy} \rangle = 0, \tag{3.4}$$

$$\langle B_{ijmn}(\mathbf{y})N^0_{m,ny}N^{kl}_{m,ny} + t_{ij}(\mathbf{y})N^{kl}_{i,jy} \rangle = 0. \tag{3.5}$$

From Eqs. (3.3) and (3.4) and the symmetry $B_{ijmn} = B_{mnij}$ (see Section 1) we obtain Eq. (3.3). This formula is the analog of the expression of the homogenized thermoelastic constants through the solution of the elastic cellular problem, see Kolpakov (1980).

If $c_{ijkl}(\mathbf{y}) = \text{const}$ then $T_{ij} = \langle t_{ij} \rangle$. Indeed, in this case from (1.21) we have $\boldsymbol{N}^{ij} = 0$ and we obtain $T_{ij} = \langle t_{ij} \rangle$ from the last formula in (3.3). It means that the initial stresses can influence the thermoelastic constants only in the case of inhomogeneous materials.

4. Laminated thermoelastic medium with the initial stresses

The above developed mathematical apparatus can be successfully applied to the calculation of the effective characteristics of the laminated media.

If we assume that the layers are parallel to the Ox_1x_2-plane; then all the properties of the material will depend only on the coordinate y_3, and the local problems will become ordinary differential equations that can be solved explicitly. In the case under consideration the cellular problem (2.3) is transformed to the following problem:

$$(c_{i3m3}(y_3)N^{kl}_{m'} + c_{i3kl}(y_3))' = 0 \quad \text{on } [0, 1], \tag{4.1}$$

$\mathbf{N}^0(y_3)$ is periodic with period 1 and the cellular problem (2.2) is transformed to

$$(c_{i3m3}(y_3)N^{0\prime}_m + t_{i3}(y_3))' = 0 \quad \text{on } [0, 1], \tag{4.2}$$

$\mathbf{N}^0(y_3)$ is periodic with period 1.

The prime means $\mathrm{d}/\mathrm{d}y_3$. Note that $\mathbf{N}^{kl}(y_3) = \boldsymbol{N}^{0kl}(y_3)$ and $\mathbf{N}^0(y_3) = \boldsymbol{N}^{00}(y_3)$.

In the considered case (2.7) takes the form

$$T_{ij}(\sigma) = T_{ij}(0) + \mu n_{ij}(\sigma), \tag{4.3}$$

where

$$n_{ij}(\sigma) = \langle \sigma^{\varepsilon}_{33}(0) N^{0\prime}_p N^{ij\prime}_p \rangle. \tag{4.4}$$

The expressions for the homogenized thermoelastic constants $T_{ij}(0)$ of composite with no initial stresses can be found in Kalamkarov and Kolpakov (1997).

It is known that in the laminated media described above $\sigma^{\varepsilon}_{33}(0) = \sigma_{33}(0)$ (see (1.8)). Then (4.4) takes the form

$$n_{ij}(\sigma) = \sigma_{33}(0) \langle N^{0\prime}_p N^{ij\prime}_p \rangle. \tag{4.5}$$

Solving the problem (4.2) we obtain

$$N^{0\prime}_i = -c_{i3n3}(y_3)^{-1} t_{n3}(y_3) + c_{i3n3}(y_3)^{-1} C_n, \tag{4.6}$$

where "−1" denotes the inverse matrix and $\{C_n\}$ are constants.

From the periodicity condition in (4.2) and (4.6) we get

$$\langle -c_{i3n3}(y_3)^{-1} t_{n3}(y_3) \rangle + \langle c_{i3n3}(y_3)^{-1} C_n \rangle = 0.$$

Solving this equation with respect to constants $\{C_i\}$, we obtain

$$C_i = \langle c_{i3n3}(y_3)^{-1} \rangle^{-1} \langle c_{n3m3}(y_3)^{-1} t_{m3}(y_3) \rangle.$$

Substituting this equality into Eq. (4.6), we get

$$N^{0\prime}_i = -c_{i3n3}(y_3)^{-1} t_{n3}(y_3) + c_{i3n3}(y_3)^{-1} \langle c_{n3m3}(y_3)^{-1} \rangle^{-1} \langle c_{m3p3}(y_3)^{-1} t_{p3}(y_3) \rangle. \tag{4.7}$$

In the similar way we obtain from (4.1)

$$N^{kl\prime}_i = -c_{i3n3}(y_3)^{-1} c_{n3kl}(y_3) + c_{i3n3}(y_3)^{-1} \langle c_{n3m3}(y_3)^{-1} \rangle^{-1} \langle c_{m3p3}(y_3)^{-1} c_{p3kl}(y_3) \rangle. \tag{4.8}$$

For isotopic materials $c_{i3k3} = \delta_{ik} c_{i3i3}$ and $t_{ij} = \delta_{ij}\alpha$, where α means the linear thermal expansion coefficient. Then $c^{-1}_{i3n3} = \delta_{in}/c_{i3i3}$, and Eq. (4.7) takes the form

$$N^{0\prime}_i = \delta_{i3} \left(\alpha(y_3) - \frac{1}{c_{3333}(y_3)} \left\langle \frac{1}{c_{3333}(y_3)} \right\rangle^{1} \langle \alpha(y_3) \rangle \right). \tag{4.9}$$

In the similar way we obtain from (4.8)

$$N^{kl\prime}_i = -\frac{c_{i3kl}(y_3)}{c_{i3i3}(y_3)} + \frac{1}{c_{i3i3}(y_3)} \left\langle \frac{1}{c_{i3i3}(y_3)} \right\rangle^{-1} \left\langle \frac{c_{i3kl}(y_3)}{c_{i3i3}(y_3)} \right\rangle. \tag{4.10}$$

Substituting Eqs. (4.9) and (4.10) into (4.5), we obtain

$$\begin{aligned}
n_{kk}(\sigma) &= \sigma_{33}(0)\left\langle\left(\alpha(y_3) - \frac{1}{c_{3333}(y_3)}\left\langle\frac{1}{c_{3333}(y_3)}\right\rangle^{-1}\langle\alpha(y_3)\rangle\right)\right.\\
&\qquad\left.\times\left(-\frac{c_{33kk}(y_3)}{c_{3333}(y_3)} + \frac{1}{c_{3333}(y_3)}\left\langle\frac{1}{c_{3333}(y_3)}\right\rangle^{-1}\left\langle\frac{c_{33kk}(y_3)}{c_{3333}(y_3)}\right\rangle\right)\right\rangle\\
&= \sigma_{33}(0)\left[-\left\langle\frac{\alpha(y_3)}{c_{33kk}(y_3)}c_{3333}(y_3)\right\rangle + \left\langle\frac{c_{33kk}(y_3)}{c_{3333}(y_3)^2}\right\rangle\left\langle\frac{1}{c_{3333}(y_3)}\right\rangle^{-1}\langle\alpha(y_3)\rangle\right.\\
&\qquad + \left\langle\frac{\alpha(y_3)}{c_{3333}(y_3)}\right\rangle\left\langle\frac{1}{c_{3333}(y_3)}\right\rangle^{-1}\left\langle\frac{c_{33kk}(y_3)}{c_{3333}(y_3)}\right\rangle\\
&\qquad\left. - \left\langle\frac{1}{c_{3333}(y_3)^2}\right\rangle\left\langle\frac{1}{c_{3333}(y_3)}\right\rangle^{-2}\langle\alpha(y_3)\rangle\left\langle\frac{c_{33kk}(y_3)}{c_{3333}(y_3)}\right\rangle\right]. \qquad (4.11)
\end{aligned}$$

By using the well-known formulas for the elastic constants of isotropic material (see e.g., Timoshenko and Goodier, 1970),

$$c_{3311} = \frac{Ev}{(1+v)(1-2v)}, \quad c_{3333} = \frac{E(1-v)}{(1+v)(1-2v)} \qquad (4.12)$$

and substituting (4.12) into (4.11), we obtain for $k = 1$

$$\begin{aligned}
n_{11}(\sigma) &= \sigma_{33}(0)\left[-\left\langle\frac{\alpha(y_3)}{\chi(y_3)}\right\rangle - \left\langle\frac{1}{\xi(y_3)\chi(y_3)E(y_3)}\right\rangle\left\langle\frac{1}{\xi(y_3)E(y_3)}\right\rangle^{-1}\langle\alpha(y_3)\rangle\right.\\
&\qquad\left. + \left\langle\frac{\alpha(y_3)}{\xi(y_3)E(y_3)}\right\rangle\left\langle\frac{1}{\xi(y_3)E(y_3)}\right\rangle^{-1}\left\langle\frac{1}{\chi}\right\rangle - \left\langle\frac{1}{\xi(y_3)E(y_3)^2}\right\rangle\left\langle\frac{1}{\xi(y_3)E(y_3)}\right\rangle^{-2}\langle\alpha(y_3)\rangle\left\langle\frac{1}{\chi(y_3)}\right\rangle\right].
\end{aligned}$$

In the considered case $n_{22}(\sigma) = n_{11}(\sigma)$.

For $k = 3$ we obtain

$$n_{33}(\sigma) = \sigma_{33}(0)\left[\left\langle\frac{\alpha(y_3)}{\xi(y_3)E(y_3)}\right\rangle\left\langle\frac{1}{\xi(y_3)E(y_3)}\right\rangle^{-1} - \left\langle\frac{1}{\xi(y_3)^2E(y_3)^2}\right\rangle\left\langle\frac{1}{\xi(y_3)E(y_3)}\right\rangle^{-2}\langle\alpha(y_3)\rangle\right],$$

where

$$\xi = \frac{1-v}{(1+v)(1-2v)}, \quad \chi = \frac{1-v}{v} \quad \text{and} \quad c_{3333} = \xi E, \quad \frac{c_{3311}(y_3)}{c_{3333}(y_3)} = \frac{1}{\chi}.$$

For $v = \text{const.}$ we get

$$n_{11}(\sigma) = n_{22}(\sigma) = \sigma_{33}(0)\left[\left\langle\frac{\alpha(y_3)}{E(y_3)}\right\rangle\left\langle\frac{1}{E(y_3)}\right\rangle^{-1} - \left\langle\frac{1}{E(y_3)^2}\right\rangle\left\langle\frac{1}{E(y_3)}\right\rangle^{-2}\langle\alpha(y_3)\rangle\right]\frac{1}{\chi},$$

$$n_{33}(\sigma) = \sigma_{33}(0)\left[\left\langle\frac{\alpha(y_3)}{E(y_3)}\right\rangle\left\langle\frac{1}{E(y_3)}\right\rangle^{-1} - \left\langle\frac{1}{E(y_3)^2}\right\rangle\left\langle\frac{1}{E(y_3)}\right\rangle^{-2}\langle\alpha(y_3)\rangle\right].$$

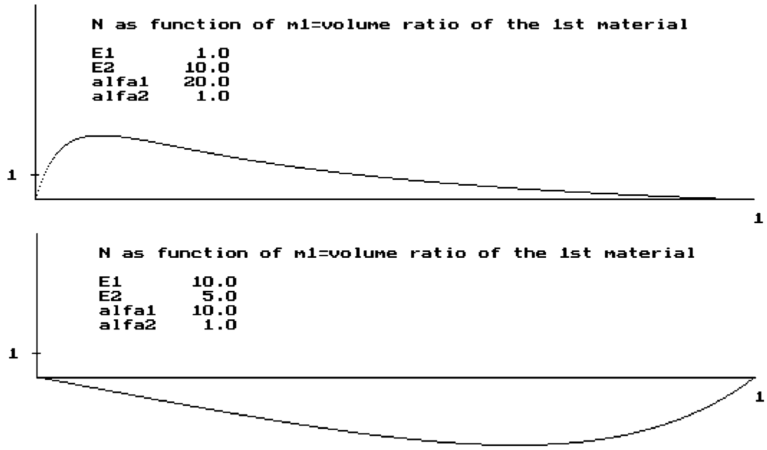

Fig. 2. N as function of μ_1.

The quantity

$$N = \left\langle \frac{\alpha(y_3)}{E(y_3)} \right\rangle \left\langle \frac{1}{E(y_3)} \right\rangle^{-1} - \left\langle \frac{1}{E(y_3)^2} \right\rangle \left\langle \frac{1}{E(y_3)} \right\rangle^{-2} \langle \alpha(y_3) \rangle$$

was computed for two-components composite. For two-component composite

$$\langle f \rangle = f_1 \mu_1 + f_2 \mu_2,$$

where μ_1 is the volume contents of the first material, $\mu_2 = 1 - \mu_1$ is the volume contents of the second material. Fig. 2 shows two plots for N as a function of μ_1. It is seen that function N can take both positive and negative values.

5. Conclusions

The homogenization problem for the inhomogeneous (composite) thermoelastic solid with the initial stresses is analyzed. It is shown that for the inhomogeneous body the asymptotic homogenization method should be applied directly to the original problem formulation in order to take into account the initial stresses in a correct way. Application of the direct analog of the classical theory to inhomogeneous bodies, in general, leads to the wrong results.

It is proved that the effective (homogenized) thermoelastic characteristics of the composite material depend not only on local distributions of the constituent materials: i.e. local elastic properties, local thermoelastic properties, but also on local initial stresses. Therefore it is shown that for the inhomogeneous (composite) material local initial stresses contribute towards the values of the effective characteristics of the material. This kind of interaction is not possible for the homogeneous materials. From the mathematical viewpoint, the asymptotic homogenization procedure is equivalent to the computation of G-limit of the corresponding operator. And the above noted mechanical phenomenon is based on the fact that in the considering case the G-limit of a sum is not equal to the sum of G-limits.

The developed general homogenized model is applied to the practically important case of the small initial stresses as compared with the magnitudes of the elastic constants. In this case the effective characteristics, both elastic and thermoelastic, of the composite with initial stresses are represented in the following form:

the coefficient corresponding to the composite with no initial stresses + the first order corrector.

It is shown that the first order correctors can be expressed through the solutions of the unit cell problem for the body with no initial stresses.

Finally, the explicit formulas for the effective thermoelastic characteristics and numerical results are obtained for a laminated composite solid with the initial stresses.

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Appendix A

A.1. Transformation of the formula (1.26) to a bilinear functional

Let us obtain a representation of the homogenized thermoelastic constants in the form of a bilinear functional. Being of interest by itself, such kind of representation is useful for the analysis of homogenization problems.

Multiply Eq. (1.21) by N_j^0 and integrate by parts over the periodicity cell Y. As a result, we obtain the following equality with account of periodicity of $\boldsymbol{N}^{kl}$ and $\boldsymbol{N}^0$:

$$\langle B_{ijmn}(\mathbf{y})N^{kl}_{m,ny}N^0_{i,jy} + B_{ijkl}(\mathbf{x},\mathbf{y})N^0_{i,jy}\rangle = 0.$$

Then taking into account that $B_{ijkl} = B_{klij}$ and changing subscripts, we obtain

$$\langle B_{ijmn}(\mathbf{x},\mathbf{y})N^0_{m,ny}\rangle = -\langle B_{klmn}(\mathbf{y})N^{ij}_{m,ny}N^0_{k,ly}\rangle. \tag{A.1}$$

Eqs. (1.26) and (A.1) yield the following formula:

$$T_{ij}(\sigma) = \langle t_{ij} - B_{klmn}(\mathbf{y})N^{ij}_{m,ny}N^0_{k,ly}\rangle. \tag{A.2}$$

The right-hand side of Eq. (A.2) is the bilinear functional.

A.2. Elimination of the functions N^{10} and N^{1ij} from the formula for the first-order corrector

By resorting to problem (2.4) and (2.3) we can eliminate the functions $\boldsymbol{N}^{10}$ and $\boldsymbol{N}^{1ij}$ from Eq. (2.6).

Multiplying Eq. (2.4) by N_i^{00} and integrating the result by parts over the unit cell Y, and taking into account the periodicity of N^{0ij} and $\boldsymbol{N}^{10}$ and the boundary condition (2.4) we obtain

$$\langle c_{ijnm}(\mathbf{y})N^{1kl}_{m,ny}N^{00}_{i,jy} + b_{ijkl}(\mathbf{x},\mathbf{y})N^{00}_{i,jy} + b_{ijmn}(\mathbf{x},\mathbf{y})N^{0kl}_{m,ny}N^{00}_{i,jy}\rangle = 0.$$

Afterwords,

$$\langle c_{ijmn}(\mathbf{y})N^{1kl}_{m,ny}N^{00}_{i,jy}\rangle = -\langle b_{klmn}(\mathbf{x},\mathbf{y})N^{0ij}_{m,ny}N^{00}_{k,ly} + b_{ijkl}(\mathbf{x},\mathbf{y})N^{00}_{i,jy}\rangle. \tag{A.3}$$

Multiplying Eq. (2.3) by N_i^{0kl} and integrating the result by parts over the cell Y, and taking into account the periodicity of N^{0ij} and $\boldsymbol{N}^{10}$ we obtain

$$\langle c_{ijmn}(\mathbf{y})N^{10}_{m,ny}N^{0kl}_{i,jy} + b_{ijmn}(\mathbf{x},\mathbf{y})N^{00}_{m,ny}N^{0kl}_{i,jy}\rangle = 0.$$

Afterwords,

$$\langle c_{ijmn}(\mathbf{y})N^{10}_{m,ny}N^{0kl}_{i,jy}\rangle = -\langle b_{ijmn}(\mathbf{x},\mathbf{y})N^{00}_{m,ny}N^{0kl}_{i,jy}\rangle. \tag{A.4}$$

Substituting Eqs. (A.3) and (A.4) into (2.6), we obtain

$$\begin{aligned} n_{ij}(\sigma) &= \langle -b_{klmn}(\mathbf{y})N^{0ij}_{m,ny}N^{00}_{k,ly} + t_{klmn}(\mathbf{y})N^{0ij}_{m,ny}N^{00}_{k,ly} + b_{klij}(\mathbf{y})N^{00}_{k,ly} + b_{klmn}(\mathbf{y})N^{00}_{m,ny}N^{0ij}_{k,ly}\rangle \\ &= \langle b_{klij}(\mathbf{y})N^{00}_{k,ly} + b_{klmn}(\mathbf{y})N^{00}_{m,ny}N^{0ij}_{k,ly}\rangle. \end{aligned}$$

Statement 1. Let the stresses σ^*_{ij} be periodic in $\mathbf{y}$ with the periodicity cell Y, and let them satisfy the equation $\sigma^*_{ij,jy} = 0$ in Y. Then $\langle \sigma^*_{ij}Z_{,jy}\rangle = 0$ for any function $Z(\mathbf{y})$ periodic in $\mathbf{y}$ with the periodicity cell Y.

In order to prove the above Statement 1, let us multiply the equation $\sigma^*_{ij,jy} = 0$ in Y, by $Z(\mathbf{y})$ and integrate the result by parts over the unit cell Y. We obtain

$$0 = \int_Y \sigma^*_{ij}Z_{,jy}\,\mathrm{d}\mathbf{y} + \int_{\partial Y} \sigma^*_{ij}Zn_j\,\mathrm{d}\mathbf{y}.$$

The second integral is equal to zero because of periodicity of $\sigma^*_{ij}(\mathbf{y})$ and $Z(\mathbf{y})$ and anti-periodicity of the vector-normal $\mathbf{n}$. The third integral is equal to zero because of the boundary condition.

Remark. By virtue of the Statement 1 the equality

$$\langle b_{ijmn}Z_{,ny}\rangle = \langle \sigma^{\varepsilon}_{jn}(0)Z_{,ny}\rangle\delta_{im} = 0 \tag{A.5}$$

takes place for any function $Z(\mathbf{y})$ periodic in y with the periodicity cell Y.

Statement 2. The initial stresses $\sigma^{\varepsilon}_{ij}(0)$ determined from the solution of elasticity problem (1.1) and (1.2) satisfy the conditions of the Statement 1.

To prove the Statement 2 we use the following well-known (see e.g., Bensoussan et al. (1978)) representation for the local stresses in the elastic body of periodic structure:

$$\sigma^{\varepsilon}_{ij}(0) = c_{ijkl}(\mathbf{y})(v_{k,lx}(\mathbf{x}) + N^{kl}_{m,ny}(\mathbf{y})v_{k,lx}(\mathbf{x})), \tag{A.6}$$

where $\mathbf{v}$ is a solution of the homogenized problem (1.6).

Using (A.6), we obtain

$$\sigma^{\varepsilon}_{ij,jy}(0) = (c_{ijkl}(\mathbf{y}) + c_{ijmn}(\mathbf{y})N^{kl}_{m,ny}(\mathbf{y})v_{m,nx}(\mathbf{x}))_{,jy}v_{k,lx}(\mathbf{x}).$$

The right-hand side of this equality is equal to zero, since it is the left-hand side of the cellular equation for the solid having no initial stresses (see (1.21)). Therefore the initial stresses $\sigma^{\varepsilon}_{ij}(0)$ determined from the solution of the elasticity problem (1.1) and (1.2) satisfy the conditions of the Statement 1.

The following equality takes place due to Statements 1 and 2:

$$\langle b_{klij}N^{00}_{k,ly}\rangle = 0. \tag{A.7}$$

The equality (A.7) can be obtained if we substitute $Z = N^0_k$ into Eq. (A.5).

From Eqs. (2.10) and (A.7) we obtain

$$n_{ij}(\sigma) = \langle b_{klmn} N^{00}_{k,ly} N^{ij}_{m,ny} \rangle = \langle b_{klmn} N^{0}_{k,ly} N^{ij}_{m,ny} \rangle, \tag{A.8}$$

where $\mathbf{N}^0 = \mathbf{N}^{00}$ and $\mathbf{N}^{ij} = \mathbf{N}^{0ij}$ are the solutions of the cellular problems with no initial stresses.

Finally, we obtain (2.7) by substituting $b_{ijmn} = \sigma^{\varepsilon}_{jn}(0)\delta_{im}$ and taking into account Eq. (A.1).

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